What is complex analysis? The main object of study is an analytic $f: \mathbb{C} \to \mathbb{C}$. As a set, $\mathbb{C} = \mathbb{R}^2$ so you may naively think that the theory is similar to real analysis. Surprisingly, the requirement of analycity, namely, that the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite on an open set, produces results that have no counterpart in the real case. The difference is that the numbers in the expression are complex.

An example of a theorem we will prove: Louivilles Theorem

Every bounded analytic function on $\mathbb{C}$ is constant. //

Chapter 1: Complex Numbers

In this chapter, we set the stage for doing complex analysis.

Main Topics:

(1) Construct the field of complex numbers

(2) Algebraic and geometric properties

(3) Basic topological ideas of $\mathbb{C}$

Let $\mathbb{R}$ be the field of real numbers. The equation

$$x^2 + 1 = 0 \quad (*)$$

has no real solutions. We seek a field $\mathbb{C}$ containing $\mathbb{R}$ that extends the operations $+, \cdot$ of real numbers <u>and</u> contains the roots of all the polynomials. Surprisingly, the construction amounts to defining a symbol i satisfying $(*)$ and then considering all sums of the form

$$x + iy \quad , x, y \in \mathbb{R}.$$

//

Construction of the Field of Complex Numbers

$\boxed{\text{Definition (The Complex Numbers)}}$ A complex number is simply an ordered pair $z = (x,y)$ of real numbers. Thus, the set of all complex numbers is given by

$$\mathbb{C} \overset{\text{def}}{=} \mathbb{R} \times \mathbb{R} = \{(x,y) : x, y \in \mathbb{R}\}.$$

If $z = (x,y)$ is a complex number, then we write

$$\operatorname{Re} z = x \quad \text{and} \quad \operatorname{Im} z = y$$

for the real and imaginary parts of z, respectively. If $\operatorname{Re} z = 0$ and $\operatorname{Im} z \neq 0$, we say that z is purely imaginary. //

$\boxed{\text{Definition (Binary operations on } \mathbb{C})}$ Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be complex numbers. Then their sum is

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

and their product is

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$ //

$\boxed{\text{Proposition}}$ There exists a subset of $\mathbb{C}$ that is algebraically indistinguishable from $\mathbb{R}$.

Proof. Consider $A = \{(x,0) : x \in \mathbb{R}\} \subseteq \mathbb{C}$. There is a bijection

$$f : \mathbb{R} \to A \quad , \quad x \longmapsto (x,0).$$

Moreover,

$$f(x+y) = (x+y, 0) = (x,0) + (y,0) = f(x) + f(y)$$
$$f(xy) = (xy, 0) = (x,0)(y,0) = f(x) f(y)$$

According to the proposition the operations of complex addition and multiplication extend the operations of addition and multiplication of real numbers. We now identify each complex number $(x,0)$ with the corresponding real number x. By abuse of notation, we write

$$x = (x,0).$$

Now, we define the imaginary unit as follows: $i \stackrel{\text{def}}{=} (0,1)$. Then

$$i^2 = ii = (0,1)(0,1) = (-1,0) = -1.$$

Moreover, for any $z = (x,y) \in \mathbb{C}$, we see that

$$\begin{aligned} z = (x,y) &= (x,0) + (0,y) \\ &= (x,0) + (0,1)(y,0) \\ &= x + iy. \end{aligned}$$

Hence, with our new notation:

$$\boxed{\mathbb{C} = \{x+iy \;;\; x,y \in \mathbb{R}\}}$$

with the convention that $i^2 = -1$. With this notation, the sum and product are written

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

The product of complex numbers can be computed by multiplying the expressions as if they were polynomials in the variable i and using $i^2 = -1$. //

Example

$$\begin{aligned} (1+i)(1-3i) &= 1 - 3i + i - 3i^2 \\ &= 1 - 3i + i + 3 \\ &= 4 - 2i. \end{aligned}$$

The proof that this works is an exercise. //

The proof that this works is an exercise. //

Proposition (Algebraic Properties of $(\mathbb{C}, +, \cdot)$)

(1) (Additive Identity)

$$0 + z = z = z + 0 \quad \forall z \in \mathbb{C}.$$

(2) (Associativity of Addition)

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3, \quad \forall z_i \in \mathbb{C}.$$

(3) (Commutativity of Addition)

$$z_1 + z_2 = z_2 + z_2, \quad \forall z_i \in \mathbb{C}$$

(4) (Additive Inverses) For all $z \in \mathbb{C}$, there exists a complex number denoted by $-z$ such that

$$z + (-z) = 0 = (-z) + z.$$

In fact, $-z \overset{\text{def}}{=} (-1)z$.

(5) (Multiplicative Identity)

$$1 \cdot z = z = z \cdot 1, \quad \forall z \in \mathbb{C}.$$

(6) (Associativity of Multiplication)

$$z_1(z_2 z_3) = (z_1 z_2) z_3, \quad \forall z_i \in \mathbb{C}$$

(7) (Commutativity of Multiplication)

$$z_1 z_2 = z_2 z_1, \quad \forall z_i \in \mathbb{C}$$

(8) (Multiplicative Inverses) For all $z \in \mathbb{C} \setminus \{0\}$, there exists a complex number denoted z^{-1} such that

$$z z^{-1} = 1 = z^{-1} z.$$

In fact, if $z = x + iy$, then $z^{-1} \overset{\text{def}}{=} \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.

In fact, if $z = x + iy$, then $z^{-1} \overset{\text{def}}{=} \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$.

(9) (Distributive Law)

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3 \quad , \forall z_i \in \mathbb{C}.$$

//

Proof. Only (8). Let $z = x + iy$ be nonzero. then

$$z \cdot z^{-1} = (x + iy)\left(\frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + i\left(-\frac{xy}{x^2+y^2} + \frac{xy}{x^2+y^2}\right)$$

$$= 1.$$

■

In the language of algebra

(1) - (4) $(\mathbb{C}, +)$ is an abelian group.

(5) - (8) $(\mathbb{C} \setminus \{0\}, \cdot)$ is an abelian group.

(1) - (9) $(\mathbb{C}, +, \cdot)$ is a field.

The existence of additive and multiplicative inverses gives rise to subtraction and division of complex numbers.

Definition (subtraction / division) Let $z_1, z_2 \in \mathbb{C}$. We define subtraction and division as follows:

$$z_1 - z_2 \overset{\text{def}}{=} z_1 + (-z_2)$$

$$\frac{z_1}{z_2} \overset{\text{def}}{=} z_1 \cdot z_2^{-1} \quad , \quad z_2 \neq 0$$

//

The formula for z_2^{-1} is difficult to remember. In practice, division is computed by writing division is computed by writing

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

and multiply out the numerator and denominator. The proof is an exercise.

$\boxed{\text{Proposition (Zero-Product Property)}}$ If $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

Proof. Assume $z_1 z_2 = 0$ and that $z_1 \neq 0$. We prove that $z_2 = 0$. Since $z_1 \neq 0$, z_1^{-1} exists. Hence,

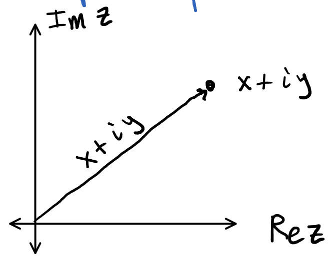
$$\begin{aligned} z_2 &= (z_1^{-1} z_1) z_2 \\ &= z_1^{-1}(z_1 z_2) \\ &= z_1^{-1} \cdot 0 \\ &= 0 \end{aligned}$$

∎

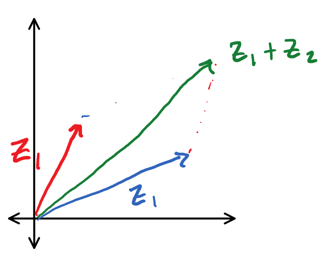
There are lots of algebraic properties in the book. Try some exercises.

Geometric Properties of Complex Numbers

As a set, $\mathbb{C} = \mathbb{R}^2$ so it is natural to visualize complex numbers as points or vectors in the complex plane



Geometrically, addition of complex numbers is just the addition of euclidean vectors



We will see a geometric interpretation of multiplication later. //

$\boxed{\text{Definition (Modulus)}}$ The modulus of a complex number $z = x+iy$ is the length of the vector (x,y), namely

$$|z| \stackrel{\text{def}}{=} \sqrt{x^2+y^2}$$

//

Notice that the modulus of a real number is just the absolute value. We can immediately derive a useful inequality:

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \geq (\operatorname{Re} z)^2 \quad (\text{or } (\operatorname{Im} z)^2)$$

Then (taking the sq. root)

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$$
$$\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|.$$

//

$\boxed{\text{Definition (Distance)}}$ The distance between two complex numbers z_1, z_2 is

$$|z_1 - z_2|.$$

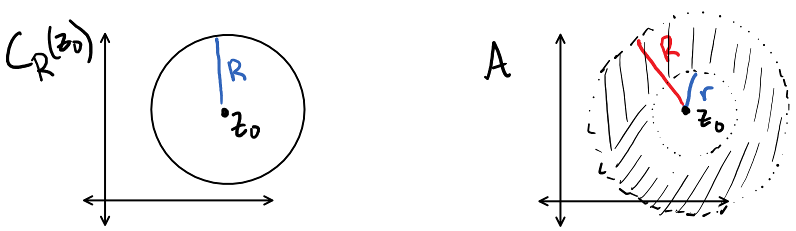
//

$\boxed{\text{Example}}$ The modulus can be used to define various subsets of $\mathbb{C}$.

(1) The circle $C_R(z_0)$ of radius $R>0$ centered at z_0 is the set

$$C_R(z_0) \stackrel{\text{def}}{=} \{ z \in \mathbb{C} : |z - z_0| = R \}.$$

(2) The open annulus of inner radius $r>0$ and outer radius $R>0$ centered at z_0 is the set $A = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$



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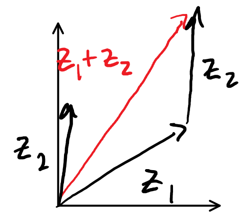
$\boxed{\text{Proposition (Triangle Inequality)}}$ For all $z_1, z_2 \in \mathbb{C}$, the following inequalities hold:

inequalities hold:



(1) $|z_1 + z_1| \leq |z_1| + |z_2|$

(2) $|z_1 + z_2| \geq \big|\, |z_1| - |z_2| \,\big|$

Proof. (1) Obvious fact about triangles.

(2) We need to show (a) $|z_1+z_2| \geq |z_1| - |z_2|$ and

(b) $|z_1 + z_2| \geq -(|z_1| - |z_2|)$.

(a)
$$\begin{aligned} |z_1| - |z_2| &= |z_1 - z_2 + z_2| - |z_2| \\ &\overset{(1)}{\leq} |z_1 + z_2| + |-z_2| - |z_2| \\ &= |z_1 + z_2| \end{aligned}$$

So this proves (a) when $|z_1| > |z_2|$. If $|z_1| < |z_2|$ switch the roles $|z_1+z_2| \geq |z_2| - |z_1| = -(|z_1| - |z_2|)$. This proves (b). $\blacksquare$

$\boxed{\text{Proposition (Modulus is Multiplicative)}}$ For all $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$,

(1) $|zw| = |z||w|$

(2) $|z^n| = |z|^n$

Proof. (1) is an easy exercise.

(2) Proof of (2) is by induction. The case $n = 2$ is just (1).

Let $n \in \mathbb{N}$. Assume $|z^n| = |z|^n$. Then

$$\begin{aligned} |z^{n+1}| &= |z^n \cdot z| \\ &\overset{(1)}{=} |z^n| \cdot |z| \\ &\overset{\text{I.H.}}{=} |z|^n \, |z| = |z|^{n+1}. \end{aligned}$$

$\blacksquare$

The following lemma is a good way to demonstrate the properties of the modulus. We will use it much later to prove the Fundamental Theorem of Algebra.

$\boxed{\text{Lemma}}$ Consider the polynomial with complex coefficients

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

There exists $R>0$ such that $|z|>R$ implies

$$\left| \frac{1}{p(z)} \right| \le \frac{2}{|a_n| R^n}.$$

(In other words, the reciprocal of a polynomial is bounded outside of a large circle $|z|=R$.)

Proof. Consider

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}, \qquad (z \neq 0).$$

Notice that $p(z) = (w + a_n) z^n$. By T.I, we get

$$|w| \le \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}.$$

Note that, for each $0 \le k \le n-1$, the term $\frac{|a_k|}{|z|^{n-k}} \longrightarrow 0$ as $|z| \longrightarrow \infty$. This means we can choose $R>0$ such that $|z|>R$, the term $\frac{|a_k|}{|z|^{n-k}} < \frac{|a_n|}{2n}$. Then $|z|>R$ implies

$$|w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$

Then $|z|>R$ implies

$$|w+a_n| \ge \big| |a_n| - |w| \big| > \left| \frac{|a_n|}{2} \right| = \frac{|a_n|}{2}.$$

Hence, $|z|>R$

$$\begin{aligned} |p(z)| &= |(w+a_n) z^n| \\ &= |w+a_n| |z|^n \\ &> \frac{|a_n|}{2} R^n. \end{aligned}$$

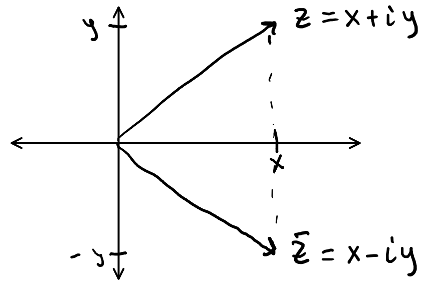
∎

Definition (Complex Conjugate) Let $z = x+iy$ be a complex number. The **complex conjugate** of z is

$$\bar{z} = x - iy.$$

//

Geometrically, $\bar{z}$ is the reflection of z about the real axis.



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Proposition (Properties of the conjugate) For all $z, w \in \mathbb{C}$:

(1) $\bar{\bar{z}} = z$

(2) $|\bar{z}| = |z|$

(3) $\overline{z+w} = \bar{z} + \bar{w}$

(4) $\overline{zw} = \bar{z}\,\bar{w}$

(5) $z\bar{z} = |z|^2$

(6) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

Proof. (1)–(3) Geometrically clear. (4),(6) Exercises.

(5) Let $z = x+iy$. Then
$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2 - ixy + iyx - i^2y^2 \\ &= x^2 + y^2 + i(yx - xy) \\ &= x^2 + y^2 \\ &= |z|^2. \end{aligned}$$

∎

Notice that (5) gives a nice formula for z^{-1}. Suppose $z \neq 0$. Then by (5) $z\bar{z} = |z|^2 \Rightarrow z\dfrac{\bar{z}}{|z|^2} = 1$. Since inverses are unique,

$$\boxed{z^{-1} = \frac{\bar{z}}{|z|^2}.}$$

//

Recall that every nonzero point $(x,y) \in \mathbb{R}^2$ can be written in polar coordinates (r, θ) where

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

This suggests the following definition.

$\boxed{\text{Definition (Polar form)}}$ If (r,θ) are polar coordinates for (x,y), then the polar form of $z = x+iy$ is

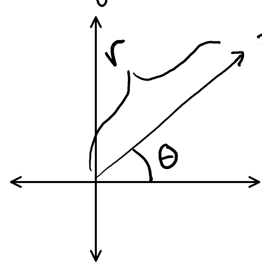
$$z = r(\cos\theta + i\sin\theta).$$

//

Evidently, r, θ are related to x, y by the equations

$$|z| = r \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

taking into account which quadrant (x,y) belongs to.



The value of θ is not unique. Each possible value is called an argument of z. The set of all possible arguments is denoted

$$\arg z.$$

The polar form is unique if we specify that $-\pi < \theta \leq \pi$. The unique argument in this interval is the principal argument $\operatorname{Arg} z$. Notice that

$$\arg z = \operatorname{Arg} z + 2k\pi \quad , \; k \in \mathbb{Z}$$

The polar form suggests a definition for the symbol $e^{i\theta}$:

$$e^{i\theta} \overset{\text{def}}{=} \cos\theta + i\sin\theta \qquad \text{(Euler's Formula)}$$

Then the polar form is written compactly in exponential form:

$$z = re^{i\theta}.$$

//

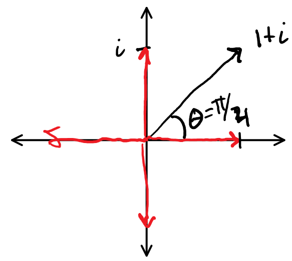
Example

(1) Exponential form of $1+i$: $|1+i| = \sqrt{2}$ and $\operatorname{Arg} 1+i = \pi/4$

So $1+i = \sqrt{2}\, e^{i\pi/4}$.

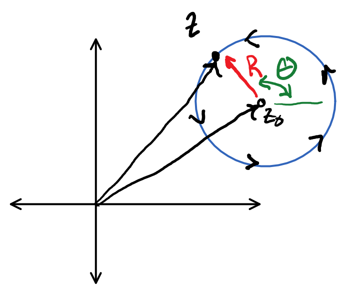
(2) $1 = e^{i0}$ $\quad i = e^{i\pi/2}$ $\quad -1 = e^{i\pi}$ $\quad -i = e^{-i\pi/2} = e^{i3\pi/2}$

Arg z $\qquad$ not Arg z



(3) The circle $C_R(z_0)$ has a nice parameterization:

$$z = z_0 + Re^{i\theta}, \quad 0 \le \theta \le 2\pi$$



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Proposition (Products / Powers in Exponential Form)

Let $z = re^{i\theta}$ and $w = se^{i\varphi}$. Then

(1) $zw = rs\, e^{i(\theta+\varphi)}$

(2) $z/w = \frac{r}{s}\, e^{i(\theta-\varphi)}$

(3) $z^{-1} = \frac{1}{r}\, e^{-i\theta}$

(4) $z^n = r^n e^{in\theta}$, $\forall n \in \mathbb{Z}$.

Proof. (1) $zw = (re^{i\theta})(se^{i\varphi})$

$= rs(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$

$= rs(\cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi))$

$$= rs(\cos\theta+\varphi \quad + i\sin\theta+\varphi)$$
$$= rs\, e^{i(\theta+\varphi)} .$$

(4) Case 1: $n \geq 1$. Proof by induction. $n=1$ is clear. Let $n \geq 1$. Assume $z^n = r^n e^{in\theta}$. Then

$$z^{n+1} = z^n \cdot z$$
$$= r^n e^{in\theta} r e^{i\theta} \overset{(1)}{=} r^{n+1} e^{i(n\theta+\theta)} = r^{n+1} e^{i(n+1)\theta} .$$

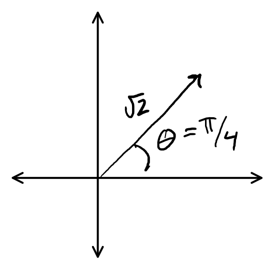
Case 2: $n < 0$. Then put $m = -n$. Apply Case 1 w/

$$z^n = (z^{-1})^m$$

Case 3: $n = 0$, by definition. ∎

$\boxed{\text{Example}}$

$$(1+i)^{2021} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{2021}$$
$$= \sqrt{2}^{2021} e^{i\left(\frac{2020\pi}{4} + \frac{\pi}{4}\right)}$$
$$= \sqrt{2}^{2020} \sqrt{2}\, e^{i\frac{2020\pi}{4}} e^{i\pi/4}$$
$$= 2^{1010} e^{505\pi} (1+i)$$
$$= -2^{1010}(1+i) .$$



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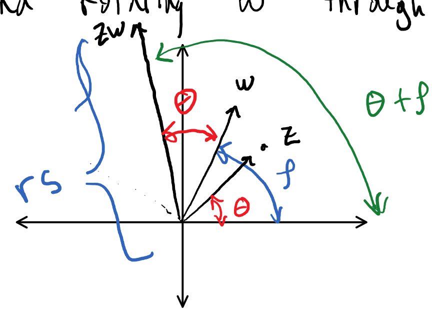
Part (1) of the proposition gives a geometric interpretation of complex multiplication. If $z = re^{i\theta}$ and $w = se^{i\varphi}$, then $zw = rs\,e^{i(\theta+\varphi)}$. This just says that zw is obtained from w by scaling w by a factor of $|z| = r$ and rotating w through an angle of $\operatorname{Arg} z$.



A couple more interesting consequences:

(1) the unit circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ is closed under multiplication.

(2) De Moivre's formula: from (4) w/ $z = e^{i\theta}$ $\Rightarrow z^n = e^{in\theta}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta .$$

//

$\boxed{\text{Proposition (Arguments of Products)}}$ Let z, w be nonzero complex, then

(1) $\arg zw = \arg z + \arg w$

(2) $\arg z/w = \arg z - \arg w$.

The statements are interpreted as follows: given the values of any 2 of the arguments, there is a value of the third satisfying the equation.

Proof. Let $z = re^{i\theta}$, $w = se^{i\varphi}$ (given values θ, φ of $\arg z / \arg w$). Then $\theta + \varphi$ is an argument of zw satisfying (1), by (4) of the proposition. Say we are given a value of $\arg zw$. Then it must be of the form

$$(\theta + \varphi) + 2k_1\pi \quad \text{for some } k_1 \in \mathbb{Z}.$$

If we are given a value of $\arg z$, it must be of the form

$$\theta + 2k_2\pi \quad \text{for some } k_2 \in \mathbb{Z}.$$

We need to find a value τ of $\arg w$ such that

$$(\theta + \varphi) + 2k_1\pi = (\theta + 2k_2\pi) + \tau.$$

Lets take $\tau = \varphi + 2(k_1 - k_2)\pi$, which is an argument of z.

Then $$(\theta + 2k_2\pi) + \tau = \theta + 2k_2\pi + \varphi + 2(k_1 - k_2)\pi$$
$$= \theta + \varphi + 2k_1\pi$$

as desired. This proves (1). Part (2) follows from (1):
$\arg \frac{z}{w} = \arg zw^{-1} = \arg z + \arg w^{-1}$. Since $w^{-1} = \frac{1}{r}e^{-i\theta}$, it is clear that $\arg w^{-1} = -\arg w$, proving the claim. $\square$

$\boxed{\text{Example}}$

(1) The principal argument of $z = (\sqrt{3} - i)^6$. An argument of $\sqrt{3} - i$ is $-\pi/6$. By the proposition

$$\arg(\sqrt{3} - i)^6 = 6 \arg \sqrt{3} - i = 6(-\pi/6) = -\pi.$$

But this isn't the principal argument of $(\sqrt{3} - i)^6$ since it doesnt lie in the interval $(-\pi, \pi]$. So $\operatorname{Arg}(\sqrt{3}-i)^6 = \pi$.

(2) The proposition is not true when arg is replaced by Arg. A counterexample is $z = i$ and $w = -1$. Then $\operatorname{Arg} z = \pi/2$, $\operatorname{Arg} w = \pi$, but $\operatorname{Arg} zw = \operatorname{Arg} -i = -\pi/2$. But $\operatorname{Arg} z + \operatorname{Arg} w = 3\pi/2$. //

Roots of Complex Numbers

Lemma Two nonzero complex numbers z, w are equal if and only if $|z| = |w|$ and $\arg z = \arg w$.

Definition (Roots) Let w be a nonzero complex number. An n^{th} root of w is a solution to $z^n = w$.

The set of all nth roots of w is denoted by $w^{1/n}$. The symbol $\sqrt[n]{}$ is reserved to denote the unique positive nth root of a positive real number. //

Proposition (Distinct roots) There are precisely n distinct nth roots of w. Namely,

$$c_k = \sqrt[n]{|w|}\, e^{i\left(\frac{\operatorname{Arg} w}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \dots, n-1.$$

Proof. Let $z = re^{i\theta}$ and $w = |w|e^{i\operatorname{Arg} w}$. We solve

$$r^n e^{in\theta} = z^n = w = |w| e^{i\operatorname{Arg} w}.$$

By the lemma, these are equal iff $r^n = |w|$ and $n\theta = \operatorname{Arg} w + 2k\pi$ where k is any integer. Hence,

$$z = re^{i\theta} = \sqrt[n]{|w|}\, e^{i\left(\frac{\operatorname{Arg} w}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbb{Z}.$$

We obtain all unique n^{th} roots by taking $K = 0, 1, \dots, n-1$ since $\sum_{k=0}^{n-1} \frac{2k\pi}{n} = 2\pi$.

∎

With the notation of the proposition, the principal root of w is

$$c_0 = \sqrt[n]{|w|}\, e^{i\frac{\operatorname{Arg} w}{n}}.$$

If we introduce the notation $\omega_n = e^{i\frac{2\pi}{n}}$, then we can write

$$\omega_n^{k} = e^{i\frac{2k\pi}{n}}$$

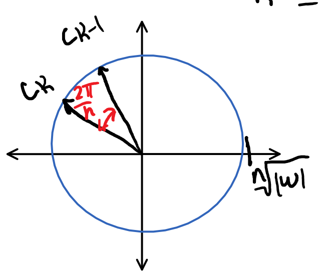
According to the proposition, the complex numbers

$$1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1} \qquad \text{(c.f. Problem Set 1 P3)}$$

are the distinct solutions to $z^n = 1$, the nth roots of unity.

We can always write the roots of w in terms of the principal root and the roots of unity:

$$c_k = \underbrace{\sqrt[n]{|w|}\, e^{i\left(\frac{\operatorname{Arg} w}{n} + \frac{2k\pi}{n}\right)}}_{c_0} = \underbrace{\sqrt[n]{|w|}\, e^{i\frac{\operatorname{Arg} w}{n}}}_{c_0}\ \underbrace{e^{\frac{2k\pi}{n}}}_{\omega_n^k}$$



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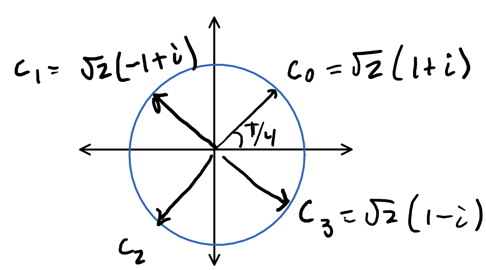
Example

(1) We compute explicitly the 4^{th} roots of -16.

$$c_k = \sqrt[4]{16}\, e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$$

$$= 2\, e^{i\pi/4}\, e^{\frac{k\pi}{2}}$$

So $c_0 = 2e^{i\pi/4} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(1+i)$.

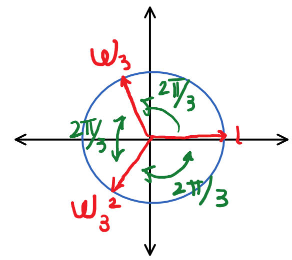


Then $c_1 = \sqrt{2}(-1+i)$, $c_2 = \sqrt{2}(-1-i)$, $c_3 = \sqrt{2}(1-i)$

(2) We compute explicitly the 3^{rd} roots of unity.

The third roots of unity are

$$1,\ \omega_3,\ \omega_3^2$$



where $\omega_3 = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Then $\omega_2 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

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Basic Topology of $\mathbb{C}$

The final topic of Ch 1 is an introduction to the basic topological ideas. The purpose is to define the kind of subsets of $\mathbb{C}$ that are suitable for doing complex analysis, namely:

nonempty open connected sets.

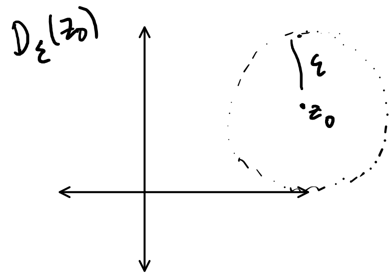
//

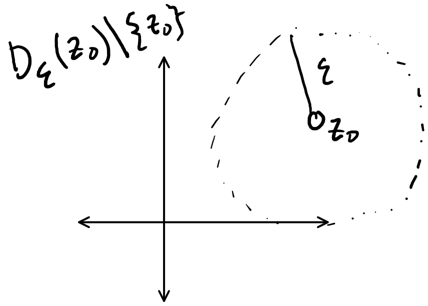
Definition (open disk / neighborhood) Let $\varepsilon > 0$. The open disk (of radius ε centered at z_0) is the set

$$D_\varepsilon(z_0) = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}.$$

We may refer to such an open disk as a neighborhood or ε-neighborhood of z_0. A deleted open disk / neighborhood is a set of the form

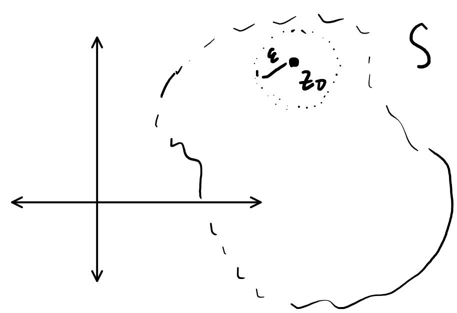
$$D_\varepsilon(z_0) \setminus \{z_0\}$$





Points within the same ε-neighborhood are "close" in the sense that they are within a distance of 2ε from eachother.

Definition (Interior Point) Let $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is an *interior point* of S if $\exists\, \varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq S$



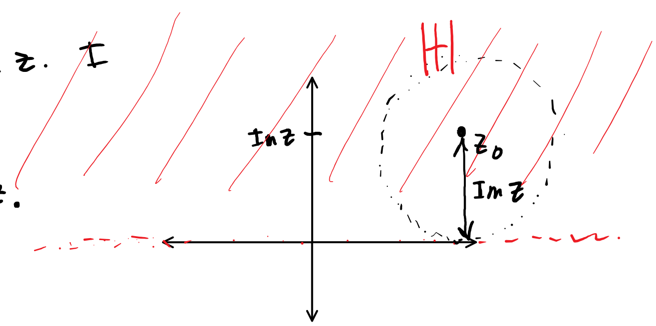
Example The *open upper half-plane*

$$\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$$

consists entirely of interior points.

Proof. Let $z \in \mathbb{H}$. Then $\operatorname{Im} z > 0$. Put $\varepsilon = \operatorname{Im} z$. I need to show

$$D_\varepsilon(z) \subseteq \mathbb{H}.$$



Let $w \in D_\varepsilon(\ )$. Then $|w - z| < \varepsilon = \operatorname{Im} z$.

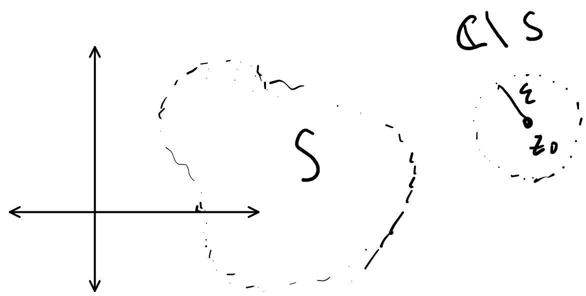
$$\operatorname{Im} z > |w - z| \geq |\operatorname{Im}(w - z)|$$
$$= |\operatorname{Im} w - \operatorname{Im} z|$$

This says

$$-\operatorname{Im} z < \operatorname{Im} w - \operatorname{Im} z < \operatorname{Im} z$$
$$\Rightarrow \quad 0 < \operatorname{Im} w \text{ . So } w \in \mathbb{H}.$$

∎

Definition (Exterior Point) Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C} \setminus S$ (the complement of S) is an exterior point if it is interior to $\mathbb{C} \setminus S = \{z_0 \in \mathbb{C},\ z_0 \notin S\}$.



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Example The points exterior to $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are the points z such that $\operatorname{Im} z < 0$.

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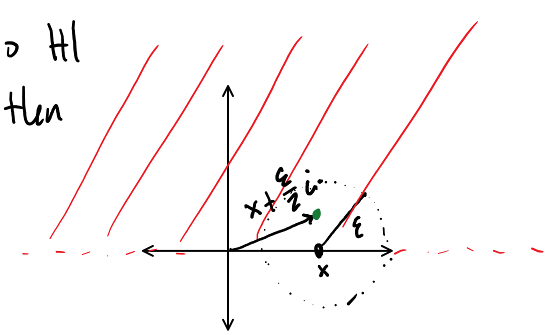
Definition (Boundary Point) Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is a boundary point of S if it is not an interior point to S and is not an exterior point to S. That is, every open disk contains points in S and not in S.

//

notation

Example The boundary $\partial \mathbb{H}$ of $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the real line.

Proof. Any point w/ $\operatorname{Im} z > 0$ is interior to $\mathbb{H}$ so not a boundary pt. Also, if $\operatorname{Im} z < 0$, then z is exterior, so not a boundary point.
So let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Note that $x \notin \mathbb{H}$. Consider $x + \frac{\varepsilon}{2} i$. Then



$$|x + \tfrac{\varepsilon}{2} i - x| = |\tfrac{\varepsilon}{2} i| = \tfrac{\varepsilon}{2} < \varepsilon.$$ So $x + \frac{\varepsilon}{2} i \in D_\varepsilon(x)$. ■

Definition (Open / closed / closure) Let $S \subseteq \mathbb{C}$.

(1) S is open if it contains none of its boundary points.

(2) S is closed if it contains all of its boundary points.

(3) The closure $\bar{S}$ of S is $\bar{S} = S \cup \partial S$.

//

Example The set $\mathbb{H}$ is open. The complement of $\mathbb{H}$ is closed. The closure of $\mathbb{H}$ is $\bar{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$. The set $\mathbb{H} \cup \{0\}$ is not open or closed.

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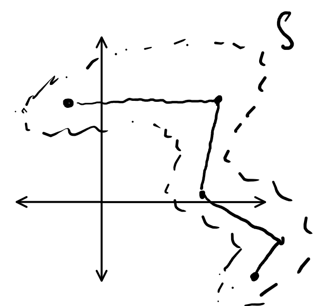
Definition (connected sets) Let S be an open set. Then S is connected if any two points in S can be connected by a polygonal line consisting of a finite number of line segments joined end to end and lying totally in the set.



A domain is nonempty, open, connected set. A region is a domain together w/ some or all of its boundary points. //

Example $\mathbb{H}$ is a domain since it is nonempty, open, and any two points in $\mathbb{H}$ can be connected by a straight line.
An example of a region is $\mathbb{H} \cup \{0\}$.

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Domains and regions are the sets we will find most suitable for stating elegant results about functions of a complex variable.

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Definition (Bounded Sets) A subset $S \subseteq \mathbb{C}$ is bounded if there exists $\varepsilon > 0$ such that $S \subseteq D_\varepsilon(0)$.

Example $\mathbb{H}$ is unbounded. //